April 2014 – Questions

From Subject CT8

1 Outline the key findings in behavioural finance. [10]

2 (i) State the expression for the return on a security, i, in the single-index model, defining all terms used. [2]

(ii) Explain the difference between the single-index model and the Capital Asset Pricing Model. [1]

Suppose the market has expected return 6% and standard deviation 10%. Two securities have expected returns 8% and 10%, and standard deviations 15% and 20%. The correlation between these two securities and the market is 0.25 and 0.4 respectively. Assume the single-index model described in part (i) holds.

(iii) Calculate the constant parameters in the expression for the return of these two securities. [5]

(iv) Explain how a multi-index model would be expected to perform relative to the single-index model, in terms of fitting data and predicting future security price moves. [2]
[Total 10]

3 Let W be a standard Brownian motion.

(i) State the continuous-time log-normal model of a security price S, defining all the terms used. [2]

Let f be a function of t and W_t^2.

(ii) (a) Find a function f such that $f(t, W_t^2)$ is a F_t- martingale, with F the Brownian filtration.

Hint: $E(W_t^2 | \mathcal{F}_s) = W_s^2 + t - s$ for all $t \geq s$.

(b) Use Ito's Lemma to show that $f(t, W_t^2)$ is a process with zero drift. [4]

Let X be the process defined as $X_t = t^{\alpha} W_{t^{\beta}}$.

(iii) Derive the values of α and β for which X_t defines a standard Brownian motion. [6]
[Total 12]

4 Consider the following long position in European and American call options written on a stock, with strikes and times to expiry as set out in the table below.

Option	*European/American*	*Strike price*	*Time to expiry*
A	American	400	3 years
B	American	400	2 years
C	American	420	3 years
D	European	400	3 years
E	European	400	2 years

Rank these options in order of value to the extent that this is possible. [5]

5 Consider the following model for the short rate r :

$$dr_t = \mu r_t dt + \sigma dZ_t$$

where μ and σ are fixed parameters and Z is a standard Brownian motion.

(i) Comment on the suitability of this model for the short rate. [4]

An alternative model for the short rate is the Vasicek model:

$$dr_t = a(\mu - r_t)dt + \sigma dZ_t .$$

(ii) Derive an expression for $\int_t^T r(u)du$. [6]

(iii) State the distribution of $\int_t^T r(u)du$. [1]

[Total 11]

6 (i) State the equation for the capital market line in the Capital Asset Pricing Model (CAPM), defining all the terms used. [3]

In a market where the CAPM is assumed to hold, the expected annual return on the market portfolio is 12%, the variance is 4%% and the effective risk-free annual rate is 4%. An Agent wants an expected annual return of 18% on a portfolio worth £1,200,000.

(ii) Calculate the standard deviation of the return on the corresponding efficient portfolio. [2]

(iii) Calculate the amount of money invested in each component of the Agent's portfolio. [3]

[Total 8]

7 In a Black-Scholes market, let S be the price of a stock and D be the price of a derivative written on S, with maturity T, where $D_t = g(t, S_t)$ for any $t < T$ and $g(T, x) = f(x)$.

(i) Write down the partial differential equation (PDE) that g must satisfy, including the boundary condition for time T. [3]

Suppose that the derivative pays ${S_T}^n / {S_0}^{n-1}$ at time T, where n is an integer greater than 1.

(ii) Show, using part (i), that the price of the derivative at time t is given by $D_t = ({S_t}^n / {S_0}^{n-1}) e^{\mu(T-t)}$ for some μ which you should determine. [6]

[Total 9]

8 (i) State and prove the put-call parity for a stock paying no dividends. [5]

In a Black-Scholes market, a European call option on the dividend-free stock, with strike price \$120 and expiry $T = 1$ year is priced at \$10.09. The continuously-compounded risk-free rate is 2% *pa* and the stock is currently priced at \$110.

(ii) Estimate the implied volatility of the stock to the nearest 1%. [4]

A European put option on the same stock has strike price \$121 and the same maturity. An investor holds a portfolio which is long one call and short one put.

(iii) Sketch a graph of the payoff at maturity of the portfolio against the stock price. [2]

(iv) (a) Determine an upper and a lower bound on the value of the portfolio at maturity.

(b) Deduce bounds for the current put price. [3]

(v) Determine the fair price of the put. [2]

[Total 16]

9 Outline the evidence against normality assumptions in models of market returns. [4]

10 A company has two zero-coupon bonds in issue. Bond A redeems in 1 year and the current price of £100 nominal is £92.50. Bond C redeems in 2 years and the current price of £100 nominal is £74.72.

The continuously-compounded risk-free rate is 2.5% *pa* for the next two years.

(i) Write down the formula for the general zero-coupon bond price in the two-state model for credit ratings, defining all the terms used. [2]

(ii) Determine the implied risk-neutral probability of default for bond A, assuming this model holds, and a recovery rate of 50% for bond A. [3]

If bond A defaults then bond C automatically defaults with a recovery rate of zero, whereas if bond A does not default then bond C may still default in the second year, but with a recovery rate of 50%.

(iii) Modify your answer to part (i) to give a formula for the current price of bond C. [3]

(iv) Calculate the risk-neutral probability of default for bond C. [3]

[Total 11]

From Subject CT6

11 Ruth takes the bus to school every morning. The bus company's ticket machine is unreliable and the amount Ruth is charged every morning can be regarded as a random variable with mean 2 and non-zero standard deviation. The bus company does offer a "value ticket" which gives a 50% discount in return for a weekly payment of 5 in advance. There are 5 days in a week and Ruth walks home each day.

Ruth's mother is worried about Ruth not having enough money to pay for her ticket and is considering three approaches to paying for bus fares:

A Give Ruth 10 at the start of each week.

B Give Ruth 2 at the start of each day.

C Buy the 50% discount card at the start of the week and then give Ruth 1 at the start of each day.

Determine the approach that will give the lowest probability of Ruth running out of money. [4]

12 A particular portfolio of insurance policies gives rise to aggregate claims which follow a Poisson process with parameter $\lambda = 25$. The distribution of individual claim amounts is as follows:

Claim	50	100	200
Probability	30%	50%	20%

The insurer initially has a surplus of 240. Premiums are paid annually in advance.

Calculate approximately the smallest premium loading such that the probability of ruin in the first year is less than 10%. [7]

13 The table below sets out incremental claims data for a portfolio of insurance policies.

Accident year	*Development year*		
	0	*1*	*2*
2011	1,403	535	142
2012	1,718	811	
2013	1,912		

Past and projected future inflation is given by the following index (measured to the mid point of the relevant year).

Year	*Index*
2011	100
2012	107
2013	110
2014	113
2015	117

Estimate the outstanding claims using the inflation-adjusted chain ladder technique. [9]

April 2014 – Solutions

From Subject CT8

Solution 1

> ***Overview***
>
> *This is a bookwork question on behavioural finance that can be answered straight from the Core Reading.*
>
> *Behavioural finance is covered in Chapter 3 of the Course Notes.*

Key findings in behavioural finance

Anchoring and adjustment

Anchoring is a term used to explain how people will produce estimates. They start with an initial idea of the answer ("the anchor") and then adjust away from this initial anchor to arrive at their final judgement.

Thus, people base perceptions on past experience or "expert" opinion, which they amend to allow for evident differences to the current conditions. The effects of anchoring are pervasive and robust and are extremely difficult to ignore, even when people are aware of the effect and aware that the anchor is ridiculous.

Prospect theory

Prospect theory is a theory of how people make decisions when faced with risk and uncertainty. It replaces the conventional risk-averse / risk-seeking decreasing marginal utility theory based on total wealth with a concept of value defined in terms of gains and losses relative to a reference point.

This generates utility curves with a point of inflexion at the chosen reference point.

Prospect theory is therefore associated with the concept of framing.

Framing (and question wording)

The way a choice is presented ("framed") and, particularly, the wording of a question in terms of gains and losses, can have an enormous impact on the answer given or the decision made. Changes in the way a question is framed of only a word or two can have a profound effect.

Myopic loss aversion

This is similar to prospect theory, but considers repeated choices rather than a single "gamble".

Research suggests that investors are less "risk-averse" when faced with a multi-period series of "gambles", and that the frequency of choice / length of reporting period will also be influential.

Overconfidence

People tend to overestimate their own abilities, knowledge and skills.

Moreover, studies show that the discrepancy between accuracy and overconfidence increases (in all but the simplest tasks) as the respondent is more knowledgeable! (Accuracy increases to a modest degree but confidence increases to a much larger degree.)

This may be a result of:

- *Hindsight bias* – events that happen will be thought of as having been predictable prior to the event; events that do not happen will be thought of as having been unlikely prior to the event.
- *Confirmation bias* – people will tend to look for evidence that confirms their point of view (and will tend to dismiss evidence that does not justify it).

Solution 2

Overview

This question is largely bookwork about the single-index model for asset returns, including a comparison with other models.

The single-index model is covered in Chapter 7, which includes a comparison with the CAPM and multifactor models.

(i) ***Single-index model (SIM)***

Under the single-index model, the return on a security, i, is given by:

$$R_i = \alpha_i + \beta_i R_M + \varepsilon_i$$

where:

- R_i is the return on security *i*
- α_i, β_i are constants
- R_M is the return on the market
- ε_i is a random variable representing the component of R_i not related to the market.

(ii) ***The difference between the single-index model and CAPM***

The single-index model is purely empirical (*ie* a statistical model) and is not based on any theoretical (or economic) relationships between β_i and the other variables, as is the case with CAPM.

(iii) ***Constant parameters***

It's not clear what is meant by "the constant parameters" in this part of the question. In the Core Reading definition of the model, α_i and β_i are the only parameters referred to explicitly as constants, but the variance of the random components, V_{ε_i}, are also constants. The Examiners' Report shows that these two variances are not needed, but for completeness we work them out in the explanation box following the main answer.

We are given values for various parameters, so we want to think of formulae that relate these to at least one of the constants we're trying to find. The values for β_i can be derived by considering the covariance of returns on security i with the returns on the market. (Note that we shouldn't use a definition of beta that relies on the CAPM interpretation of beta.)

Then the α_i can be found from the expressions for the expected return on the security.

Using the definition of R_i from part (i), we can say:

$$\begin{aligned}\text{cov}(R_i, R_M) &= \text{cov}(\alpha_i + \beta_i R_M + \varepsilon_i, R_M) \\ &= \text{cov}(\alpha_i, R_M) + \beta_i \,\text{cov}(R_M, R_M) + \text{cov}(\varepsilon_i, R_M) \\ &= 0 + \beta_i V_M + 0 \\ &= \beta_i V_M\end{aligned}$$

We also know that correlation and covariance are related via the relationship:

$$\text{cov}(R_i, R_M) = \rho_i \sigma_i \sigma_M$$

Equating these two expressions for $\text{cov}(R_i, R_M)$ gives:

$$\beta_1 = \frac{\rho_1 \sigma_1 \sigma_M}{V_M} = \frac{\rho_1 \sigma_1}{\sigma_M} = \frac{0.25 \times 15}{10} = 0.375$$

$$\beta_2 = \frac{\rho_2 \sigma_2}{\sigma_M} = \frac{0.4 \times 20}{10} = 0.8$$

The expected return on security i is given by:

$$E_i = \alpha_i + \beta_i E_M$$

Rearranging this gives:

$$\alpha_i = E_i - \beta_i E_M$$

So: $\alpha_1 = E_1 - \beta_1 E_M = 8 - 0.375 \times 6 = 5.75\%$

$\alpha_2 = E_2 - \beta_2 E_M = 10 - 0.8 \times 6 = 5.2\%$

The variance of returns on security i is given by:

$$V_i = \beta_i^2 V_M + V_{\varepsilon_i}$$

Rearranging this gives:

$$V_{\varepsilon_i} = V_i - \beta_i^2 V_M$$

So: $V_{\varepsilon_1} = V_1 - \beta_1^2 V_M = 15^2 - 0.375^2 \times 10^2 = 210.94\%\% = (14.52\%)^2$

$$V_{\varepsilon_2} = V_2 - \beta_2^2 V_M = 20^2 - 0.8^2 \times 10^2 = 336\%\% = (18.33\%)^2$$

(iv) ***Multi-index model vs single-index model***

The greater number of factors used in a multi-factor model compared with the single-index model leads to a better fit to historical data.

However, correlation with the market is the most significant factor in explaining security price variation and there is little evidence that multi-factor models are significantly better at forecasting the future.

Solution 3

Overview

Part (i) asks for standard bookwork and part (ii) looks at the two different ways we can check whether a process is a martingale, which are standard techniques. Part (iii) is a little more fiddly and takes a bit of insight to find both pairs of values for α and β. The continuous-time lognormal model is defined in Chapter 11. Martingales and standard Brownian motion are defined in Chapter 9, and Ito's Lemma is covered in Chapter 10.

(i) ***Continuous-time lognormal model***

The continuous-time lognormal model of security prices states that, for $u > t$, log returns are given by:

$$\log(S_u) - \log(S_t) \sim N\left[\mu(u-t), \sigma^2(u-t)\right]$$

where μ is the *drift* of the log price, and σ is the *volatility*.

Note that the μ that appears in this definition of the lognormal model refers to the drift of the log price. This is not the same as the rate of drift of the price itself, which is $\mu + \frac{1}{2}\sigma^2$. The corresponding stochastic differential equations for the log price and the price itself are $d(\log S_t) = \mu dt + \sigma dW_t$ and $dS_t = (\mu + \frac{1}{2}\sigma^2) S_t dt + \sigma S_t dW_t$.

Solving the SDE for S_t gives $S_t = S_0 e^{\mu t + \sigma W_t}$. Since all of these specifications are equivalent, any one of them would have been acceptable in the exam.

Alternatively, it would also have been acceptable to define the model using the geometric Brownian motion SDE $dS_t = \mu S_t dt + \sigma S_t dW_t$ or its solution $S_t = S_0 e^{\left(\mu - \frac{1}{2}\sigma^2\right)t + \sigma W_t}$.

(ii)(a) ***Find f***

You may well know the required function without having to work anything out as it's one of the standard martingale functions of Brownian motion and is given on page 38 of The Tables. If not, the hint, together with the definition of a martingale, should be enough to see what the answer is.

Rearranging the hint gives:

$$E\left[W_t^2 - t \middle| F_s\right] = W_s^2 - s$$

This is the defining property of a martingale.

Assuming that the bounded condition is also satisfied, *ie* $E\left[\left|W_t{}^2 - t\right|\right] < \infty$ for all t, then the required function is:

$$f\left(t, W_t^2\right) = W_t^2 - t\,.$$

You weren't expected to prove that the bounded condition holds, but it can be shown using the triangle inequality, ie $|x+y| \le |x| + |y|$. This allows us to say, for each fixed t:

$$\begin{aligned} E\left[\left|W_t{}^2 - t\right|\right] &\le E\left[\left|W_t{}^2\right| + |-t|\right] \\ &= E\left[W_t{}^2\right] + |t| \\ &= \left\{\text{var}(W_t) + E[W_t]^2\right\} + |t| \\ &= \{t + 0\} + |t| \\ &= t + |t| < \infty \end{aligned}$$

(ii)(b) ***Use Ito's Lemma to show that f has zero drift***

The function f is defined as a function of t and W_t^2, but it's easier to treat it as a function of t and W_t. We can then apply Ito's Lemma, either implicitly by using Taylor's formula, or explicitly. We show both methods below, and then show in a separate box what the workings would have looked like if we'd treated f as a function of t and W_t^2.

Method 1 – Taylor's formula

Taylor's formula for a function of two variables, with one of them being stochastic, is:

$$df\left(t,W_t\right)=\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial W_t}dW_t+\tfrac{1}{2}\frac{\partial^2 f}{\partial W_t^2}\underbrace{\left(dW_t\right)^2}_{dt}$$

With $f\left(t,W_t\right)=W_t^2-t$, we have:

- $\dfrac{\partial f}{\partial t}=-1$
- $\dfrac{\partial f}{\partial W_t}=2W_t$
- $\dfrac{\partial^2 f}{\partial W_t^2}=2$

So: $$df\left(t,W_t\right)=\cancel{(-1)dt}+(2W_t)dW_t+\cancel{\tfrac{1}{2}(2)dt}$$

$$=2W_t dW_t$$

There is no "dt" term, *ie* the process has zero drift.

Method 2 – Ito's Lemma

In this box, we'll show how to get Ito's Lemma in the right form for use in this question part, starting from the notation given in the Core Reading and also starting from the notation in the Tables.

***Using the notation in the Core Reading**, if $dX_t=Y_t dW_t+A_t dt$, then Ito's Lemma for a function $f(t,X_t)$ is:*

$$df\left(t,X_t\right)=\frac{\partial f}{\partial X_t}Y_t dW_t+\left[\frac{\partial f}{\partial t}+\frac{\partial f}{\partial X_t}A_t+½\frac{\partial^2 f}{\partial X_t^2}Y_t^2\right]dt$$

In the particular example in this question part, $X_t=W_t$ and so we have $Y_t=1$ and $A_t=0$, ie $dW_t=1dW_t+0dt$.

With these values for Y_t and A_t, Ito's Lemma becomes:

$$df(t,W_t)=\frac{\partial f}{\partial W_t}1\,dW_t+\left[\frac{\partial f}{\partial t}+\frac{\partial f}{\partial W_t}0+½\frac{\partial^2 f}{\partial W_t^2}1^2\right]dt$$

$$=\frac{\partial f}{\partial W_t}dW_t+\left[\frac{\partial f}{\partial t}+½\frac{\partial^2 f}{\partial W_t^2}\right]dt$$

***Alternatively, using the notation on page 46 of the Tables**, if $dX_t=a\,dt+b\,dW_t$, then Ito's Lemma for a function $f(t,X_t)$ is:*

$$df(t,X_t)=\left[a\frac{\partial f}{\partial X_t}+½b^2\frac{\partial^2 f}{\partial X_t^2}+\frac{\partial f}{\partial t}\right]dt+b\frac{\partial f}{\partial X_t}dW_t$$

In the particular example in this question part, $X_t=W_t$ and so we have $a=0$ and $b=1$, ie $dW_t=0dt+1dW_t$.

With these values for a and b, Ito's Lemma becomes:

$$df(t,W_t)=\left[0\frac{\partial f}{\partial W_t}+½1^2\frac{\partial^2 f}{\partial W_t^2}+\frac{\partial f}{\partial t}\right]dt+1\frac{\partial f}{\partial W_t}dW_t$$

$$=\left[½\frac{\partial^2 f}{\partial W_t^2}+\frac{\partial f}{\partial t}\right]dt+\frac{\partial f}{\partial W_t}dW_t$$

Given the Ito process $dW_t=1dW_t+0dt$, Ito's Lemma for the function $f(t,W_t)$ is:

$$df(t,W_t)=\frac{\partial f}{\partial W_t}1dW_t+\left[\frac{\partial f}{\partial t}+\frac{\partial f}{\partial W_t}0+½\frac{\partial^2 f}{\partial W_t^2}1^2\right]dt$$

With $f(t,W_t)=W_t^2-t$, we have:

- $\frac{\partial f}{\partial t}=-1$
- $\frac{\partial f}{\partial W_t}=2W_t$
- $\frac{\partial^2 f}{\partial W_t^2}=2$

So: $$df(t,W_t)=2W_t dW_t+\left[\cancel{-1+2W_t\times 0+½\times 2\times 1^2}\right]dt$$

$$=2W_t dW_t$$

There is no "dt" term, *ie* the process has zero drift.

As an alternative application of Taylor's formula in Method 1, we can treat W_t^2 as a variable in its own right and say:

$$df\left(t,W_t^2\right)=\frac{\partial f}{\partial t}dt+\frac{\partial f}{\partial\left(W_t^2\right)}d\left(W_t^2\right)+\tfrac{1}{2}\frac{\partial^2 f}{\partial\left(W_t^2\right)^2}\left[d\left(W_t^2\right)\right]^2 \qquad (*)$$

We know $f\left(t,W_t\right)=W_t^2-t$, so the three partial derivatives are:

- $\dfrac{\partial f}{\partial t}=-1$
- $\dfrac{\partial f}{\partial\left(W_t^2\right)}=1$
- $\dfrac{\partial^2 f}{\partial\left(W_t^2\right)^2}=0$

This just leaves $d\left(W_t^2\right)$ in the middle of the right-hand side of (*) to simplify, which requires Taylor again:

$$d\left(W_t^2\right)=\frac{\partial\left(W_t^2\right)}{\partial W_t}dW_t+\tfrac{1}{2}\frac{\partial^2\left(W_t^2\right)}{\partial W_t^2}\underbrace{\left(dW_t\right)^2}_{dt}$$

$$=2W_t dW_t+\tfrac{1}{2}2dt$$

Substituting all these back into the top equation gives:

$$df\left(t,W_t^2\right)=(-1)dt+(1)\left(2W_t dW_t+dt\right)+0$$

$$=2W_t dW_t \qquad \textit{as before.}$$

(iii) ***Values for α and β***

*We could consider all the defining properties of standard Brownian motion in turn. However, it's more efficient here to make use of the fact that, since standard Brownian motion is a **Gaussian** stochastic process, we know that it is completely determined by its expectation and covariance function.*

For standard Brownian motion, we know that $E\left[W_t-W_s\right]=0$ and $\text{cov}\left(W_s,W_t\right)=\min(s,t)$, so we need to see what values of α and β give the same results for X_t.

We are given $X_t=t^{\alpha}W_{t^{\beta}}$. This is a constant, t^{α}, multiplied by a Gaussian stochastic process, $W_{t^{\beta}}$, so X_t is also a Gaussian stochastic process.

Any Gaussian stochastic process is completely determined by its expectation and covariance function.

For standard Brownian motion, W_t, we have:

$$E\left[W_t - W_s\right] = 0$$

$$\text{cov}\left(W_s, W_t\right) = \min(s,t)$$

For X_t also to be a standard Brownian motion, we need the same two results relating to the expectation and covariance to hold. Now:

$$E\left[X_t - X_s\right] = E\left[t^{\alpha} W_{t^{\beta}} - s^{\alpha} W_{s^{\beta}}\right]$$

$$= t^{\alpha} \underbrace{E\left[W_{t^{\beta}}\right]}_{0} - s^{\alpha} \underbrace{E\left[W_{s^{\beta}}\right]}_{0} = 0$$

Also:

$$\text{cov}\left(X_s, X_t\right) = \text{cov}\left(s^{\alpha} W_{s^{\beta}}, t^{\alpha} W_{t^{\beta}}\right)$$

$$= s^{\alpha} t^{\alpha} \, \text{cov}\left(W_{s^{\beta}}, W_{t^{\beta}}\right)$$

$$= s^{\alpha} t^{\alpha} \times \min\left(s^{\beta}, t^{\beta}\right)$$

$$= \min\left(s^{\alpha+\beta} t^{\alpha}, s^{\alpha} t^{\alpha+\beta}\right)$$

For this to equal $\min(s,t)$, we need:

$$\left.\begin{aligned} s^{\alpha+\beta} t^{\alpha} &= s \\ s^{\alpha} t^{\alpha+\beta} &= t \end{aligned}\right\} \Rightarrow \alpha = 0, \beta = 1$$

For this to equal $\min(t,s)$, we need:

$$\left.\begin{aligned} s^{\alpha+\beta} t^{\alpha} &= t \\ s^{\alpha} t^{\alpha+\beta} &= s \end{aligned}\right\} \Rightarrow \alpha = 1, \beta = -1$$

The first set of values just gives us $X_t = W_t$, which is clearly standard Brownian motion. The second set of values gives us $X_t = tW_{\frac{1}{t}}$, which demonstrates the time-inversion property of Brownian motion.

Solution 4

> ### *Overview*
>
> *European and American options, and the factors affecting options prices are covered in Chapter 12.*

Rank options in order of value

The value of a call option increases with time to expiry, all else being equal, so:

- A is more valuable than B
- D is more valuable than E (*unless there are* <u>*very*</u> *high dividends – see explanation box below*).

> *The longer the time to expiry, the greater the chance that the underlying stock price can increase significantly before expiry, resulting in a bigger payoff. (It could also fall significantly, but the call option holder has protection against this since they are not forced to exercise the option to buy the stock.)*
>
> *Although this result from the Core Reading is generally true, it might not be the case for European options if the dividends on the stock are VERY high. In such cases, exercising sooner in order to start receiving the high dividends might be more attractive than keeping the optionality and seeing the capital value of the stock be eroded by the dividends.*

The value of a call option increases with *decreasing* strike price, all else being equal, so:

- A is more valuable than C.

American options are at least as valuable as their European equivalent, so:

- A is at least as valuable as D
- B is at least as valuable as E.

So A is the most valuable, B is at least as valuable as E and it is impossible to rank the other pairings (although D is *probably* more valuable than E).

Solution 5

Overview

Part (i) presents a non-standard model and asks about its suitability. Provided you remember a checklist of properties to assess the model against, this should be manageable.

Part (ii) involves more tricky algebra. Solving Vasicek to get $r(t)$ is a standard piece of algebra, but integrating the result involves dealing with a double stochastic integral and this is quite challenging.

Term structure models are the topic of Chapter 18 of the Course Notes.

(i) ***Suitability of model for the short rate***

Comparing the SDE here with the SDE given later in the question for the Vasicek model we can see they match up if we set $\mu = 0$ in the Vasicek model and equate "a" in the Vasicek model with minus "μ" in the first model. This change of sign is important for model suitability as it affects the mean-reversion property.

- The model is arbitrage free, which is good.
- It allows negative interest rates, due to the volatility term having a normal distribution.
- It is only mean reverting if $\mu < 0$ and then it mean reverts to zero.
- It's easy to use to price bonds and other simple interest rate derivatives since the model is computationally tractable.
- It can't produce a realistic range of yield-curve shapes.
- With only two parameters (μ and σ), it can't fit historical data or be calibrated to current market data accurately. For example, the assumption of constant volatility isn't borne out in practice.
- Being a one-factor model, it is not flexible enough to cope with more complicated interest rate derivatives where the correlations between interest rates are important.

(ii) ***An expression for $\int_t^T r(u)du$***

The first step is to solve the Vasicek SDE to find $r(u)$. This part of the question doesn't specify a time for us to assume the short rate is known at, but looking at the integral expression, it makes sense to condition our expressions on $r(t)$. This is what we'll do in our solution, but if you conditioned on $r(0)$, you would have received equivalent credit.

The Vasicek model is an example of an Ornstein-Uhlenbeck process, so we can use the method described in Chapter 9 of the Course Notes, ie use an integrating factor of e^{at}*. This can be done by introducing a new stochastic process equal to the integrating factor times* $r(t)$*, then finding and solving the SDE for the new process.*

This solution for $r(u)$ *can then be inserted into the required integral expression.*

Let $f(t)=e^{at}r(t)$.

Then:

$$\begin{aligned} df(t) &= d\left(e^{at}r(t)\right) \\ &= d\left(e^{at}\right)r(t)+e^{at}dr(t) \\ &= ae^{at}dt\times r(t)+e^{at}\left\{a\left[\mu-r(t)\right]dt+\sigma dZ_t\right\} \\ &= \left\{\cancel{ae^{at}r(t)}+e^{at}a\left[\mu\cancel{-r(t)}\right]\right\}dt+e^{at}\sigma dZ_t \\ &= a\mu e^{at}dt+\sigma e^{at}dZ_t \end{aligned}$$

Here we've used the product rule in the second line, but we could just as well have used a Taylor/Ito approach, by considering $f(t)$ *as a function of t and* $r(t)$*.*

Relabelling the time parameter s and then integrating both sides from t to u gives:

$$\int_t^u df(s)=\int_t^u a\mu e^{as}ds+\int_t^u \sigma e^{as}dZ_s$$

$$\left[f(s)\right]_t^u=\left[\mu e^{as}\right]_t^u+\sigma\int_t^u e^{as}dZ_s$$

$$f(u)-f(t)=\mu\left(e^{au}-e^{at}\right)+\sigma\int_t^u e^{as}dZ_s$$

Now, we substitute in for f to get the equation in terms of the short rate r and rearrange to get:

$$e^{au}r(u)-e^{at}r(t)=\mu\left(e^{au}-e^{at}\right)+\sigma\int_t^u e^{as}dZ_s$$

$$r(u)=r(t)e^{-a(u-t)}+\mu\left(1-e^{-a(u-t)}\right)+\sigma\int_t^u e^{-a(u-s)}dZ_s$$

Inserting this into the required integral and splitting up the terms gives:

$$\int_t^T r(u)\,du = \int_t^T \left\{ r(t)e^{-a(u-t)} + \mu\left(1-e^{-a(u-t)}\right) + \sigma\int_t^u e^{-a(u-s)}dZ_s \right\} du$$

$$= \underbrace{r(t)\int_t^T e^{-a(u-t)}du}_{(A)} + \underbrace{\mu\int_t^T \left(1-e^{-a(u-t)}\right)du}_{(B)} + \underbrace{\sigma\int_t^T\int_t^u e^{-a(u-s)}dZ_s\,du}_{(C)}$$

This is an expression for $\int_t^T r(u)\,du$, but we haven't done enough for six marks yet, so we'll carry on and simplify the terms on the right-hand side.

(A) and (B) are deterministic integrals and so can be evaluated as:

$$(A) = r(t)\left[-\tfrac{1}{a}e^{-a(u-t)}\right]_t^T = r(t)\tfrac{1}{a}\left(1-e^{-a(T-t)}\right)$$

$$(B) = \mu\left[u+\tfrac{1}{a}e^{-a(u-t)}\right]_t^T = \mu(T-t)+\tfrac{\mu}{a}\left(e^{-a(T-t)}-1\right)$$

So $\quad (A)+(B) = \mu(T-t) + \dfrac{r(t)-\mu}{a}\left(1-e^{-a(T-t)}\right)$

(C) is a double stochastic integral and so is trickier to deal with. It's not clear whether we need to simplify (C), but it's worth thinking about if you've got spare time at the end.

By swapping the order of integration, we can create a deterministic integral in the middle of the expression, which can be evaluated to produce a single Ito integral overall.

To swap the order of integration, we first note that s varies between t and u (in the inner integral) and then u varies between t and T (in the outer integral). So algebraically we have $t \le s \le u \le T$. The other way of looking at these relationships would be to say that u varies between s and T, with s varying between t and T.

We can swap the order of integration in (C) as follows:

$$(C) = \sigma\int_t^T \left\{\int_t^u e^{-a(u-s)}dZ_s\right\} du$$

$$= \sigma\int_t^T \left\{\int_s^T e^{-a(u-s)}du\right\} dZ_s$$

$$= \sigma\int_t^T \left[-\tfrac{1}{a}e^{-a(u-s)}\right]_s^T dZ_s$$

$$= \frac{\sigma}{a}\int_t^T \left(1-e^{-a(T-s)}\right)dZ_s$$

(iii) ***The distribution of $\int_t^T r(u)du$***

The distribution of $\int_t^T r(u)du$ is determined by the term (C) from part (ii). Since $dZ_s \sim N(0,ds)$, (C) must also have a normal distribution as it can be regarded as an infinite sum of independent normally distributed terms.

The question says "state" for one mark and so no explanation is needed.

$\int_t^T r(u)du$ has a normal distribution.

We're not asked to determine the parameters of the distribution, but if we were, we could use the result from Chapter 10, which tells us that:

$$\int_0^t f(s)dB_s \sim N\left(0, \int_0^t f(s)^2 ds\right)$$

So, the mean of $\int_t^T r(u)du$ is given by $(A)+(B)$ and the variance is given by:

$$\begin{aligned}\frac{\sigma^2}{a^2}\int_t^T\left(1-e^{-a(T-s)}\right)^2 ds &= \frac{\sigma^2}{a^2}\int_t^T\left(1-2e^{-a(T-s)}+e^{-2a(T-s)}\right)ds \\ &= \frac{\sigma^2}{a^2}\left[s-\tfrac{2}{a}e^{-a(T-s)}+\tfrac{1}{2a}e^{-2a(T-s)}\right]_t^T \\ &= \frac{\sigma^2}{a^2}\left[(T-t)-\frac{2}{a}\left(1-e^{-a(T-t)}\right)+\frac{1}{2a}\left(1-e^{-2a(T-t)}\right)\right]\end{aligned}$$

Solution 6

Overview

Questions on the capital asset pricing model (CAPM) come up about every other exam paper on average. Most of these questions are calculation-based questions like this one.

CAPM and the capital market line is covered in Chapter 8 of the Course Notes.

(i) ***The capital market line***

The equation for the capital market line is:

$$E_P = r + \left(\frac{E_M - r}{\sigma_M}\right)\sigma_P$$

where:

- E_P is the expected return of any portfolio on the efficient frontier
- σ_P is the standard deviation of the return on portfolio P
- E_M is the expected return on the market portfolio
- σ_M is the standard deviation of the return on the market portfolio
- r is the risk-free rate of return.

(ii) ***Calculate σ_P***

Rearranging the equation for the capital market line gives:

$$\sigma_P = \frac{E_P - r}{E_M - r}\sigma_M = \frac{18-4}{12-4}\sqrt{4} = \frac{14}{8}2 = 3.5\%pa$$

(iii) ***Amount invested in each component of agent's portfolio***

If we assume that the agent's portfolio is to be efficient, it will be a weighted average of the risk-free asset and the market portfolio.

Let x be the proportion of the agent's efficient portfolio invested in the market portfolio (and hence $1-x$ is the proportion invested in the risk-free asset). Then the return on the agent's portfolio is given by:

$$18 = 12x + 4(1-x)$$
$$= 12x + 4 - 4x = 8x + 4$$

Solving this for x gives:

$$x = \frac{18-4}{8} = 1.75$$

So the amount of money invested in the market portfolio is:

$$£1,200,000 \times 1.75 = £2.1m$$

which requires a cash (*ie* the risk-free asset) borrowing of $£0.9m$.

Solution 7

Overview

This question is a relatively straightforward application of the PDE, although the notation used might be confusing.

The Black-Scholes PDE is covered in Chapter 15 of the Course Notes.

(i) ***PDE for g***

The Black-Scholes PDE is given on page 46 of the Tables. We simply need to replace the "f" in the Tables with the "g" that's used in this question.

The function g must satisfy:

$$\frac{\partial g}{\partial t} + (r-q)S_t \frac{\partial g}{\partial S_t} + \tfrac{1}{2}\sigma^2 S_t^2 \frac{\partial^2 g}{\partial S_t^2} = rg$$

where:

- r is the continuously-compounded risk-free rate
- q is the continuously-compounded dividend rate
- σ is the assumed volatility of the stock price over the period 0 to T.

We're not told specifically to define the notation we use that isn't defined in the question, but it's safest to do so. We're also not told whether the stock pays dividends, so again, it's safest to include them. For a non-dividend-paying stock we'd set $q = 0$.

The boundary condition applies at maturity and is:

$$D_T = g(T, S_T) = f(S_T)$$

(ii) ***Price of derivative at time t and value for μ***

A European derivative is completely specified by its payoff function at maturity and, in a Black-Scholes world, its value at earlier times t is given by the unique solution to the Black-Scholes PDE that satisfies the boundary condition.

So we need to check that the proposed formula for D_t satisfies both the boundary condition and the PDE.

Suppose $D_t = g(t, S_t) = \dfrac{S_t^n}{S_0^{n-1}} e^{\mu(T-t)}$ with $n > 1$.

Then $D_T = g(T, S_T) = f(S_T) = \dfrac{S_T^n}{S_0^{n-1}}$, so the boundary condition is satisfied.

The partial derivatives in the PDE from part (i) are given by:

$$\frac{\partial g}{\partial t} = -\mu \frac{S_t^n}{S_0^{n-1}} e^{\mu(T-t)} = -\mu g$$

$$\frac{\partial g}{\partial S_t} = \frac{nS_t^{n-1}}{S_0^{n-1}} e^{\mu(T-t)} = \frac{n}{S_t} g$$

$$\frac{\partial^2 g}{\partial S_t^2} = \frac{n(n-1)S_t^{n-2}}{S_0^{n-1}} e^{\mu(T-t)} = \frac{n(n-1)}{S_t^2} g$$

Substituting these into the PDE gives:

$$-\mu g + (r-q)S_t \frac{n}{S_t} g + \tfrac{1}{2}\sigma^2 S_t^2 \frac{n(n-1)}{S_t^2} g = rg$$

Cancelling the S_t's and the g's gives:

$$-\mu + (r-q)n + \tfrac{1}{2}\sigma^2 n(n-1) = r$$

So, for the specified D_t to be the value of the derivative at time t, we must have:

$$\mu = (r-q)n - r + \tfrac{1}{2}\sigma^2 n(n-1)$$

If we set $q = 0$, we get:

$$\mu = rn - r + \tfrac{1}{2}\sigma^2 n(n-1)$$

$$= r(n-1) + \tfrac{1}{2}\sigma^2 n(n-1)$$

$$= \left(r + \tfrac{1}{2}\sigma^2 n\right)(n-1)$$

Solution 8

Overview

Part (iv) asks about bounds and is a little tricky unless you realise it relates to the graph in part (iii).

The put-call parity relationship is derived in Chapter 12. This chapter also covers payoffs and bounds. The Black-Scholes call option pricing formula, and implied volatility in particular, is covered in Chapter 15.

(i) ***Put-call parity***

Assume that the market is arbitrage free. Consider two portfolios at time 0:

A: one call option plus cash of Ke^{-rT}

B: one put option plus one unit of stock.

where:

- the call and put option both have strike price, K, and time to expiry, T
- r is the constant, continuously-compounded risk-free rate.

Portfolio A has a payoff at expiry of:

$$\max\{S_T - K, 0\} + K = \max\{S_T, K\}$$

Portfolio B has a payoff at expiry of:

$$\max\{K - S_T, 0\} + S_T = \max\{K, S_T\}$$

Since these two payoffs are identical and there are no cashflows associated with either portfolio before expiry, then by the principle of no-arbitrage the two portfolios must also have identical values at time 0, *ie*:

$$c_0 + Ke^{-rT} = p_0 + S_0$$

This is the put-call parity relationship for a stock paying no dividends.

(ii) ***Implied volatility***

Since we're in a Black-Scholes market, we need to use the Black-Scholes pricing formula to estimate the implied volatility of the option. This may be found on page 47 of the Tables.

The Black-Scholes formula for the price at time 0 of a call option on a non-dividend-paying stock is:

$$c_0 = S_0\Phi(d_1) - Ke^{-rT}\Phi(d_2)$$

Working in dollars and substituting the known values of the parameters gives:

$$c_0 = 110\Phi(d_1) - 120e^{-0.02}\Phi(d_2)$$

We now need to find the implied volatility σ, the value of σ that makes c_t equal to the observed price of \$10.09. We need to do this by trial and interpolation. In the absence of any indications to the contrary, a good value of σ to start with (when the underlying asset is a share) is $\sigma = 0.2$, ie 20%.

If the volatility σ is assumed to be 20%, we have:

$$d_1 = \frac{\ln\frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\frac{110}{120} + (0.02 + \frac{1}{2} \times 0.2^2) \times 1}{0.2\sqrt{1}} = -0.23506$$

and $d_2 = d_1 - \sigma\sqrt{T} = -0.23506 - 0.2\sqrt{1} = -0.43506$

$$\Rightarrow \quad c_0 = 110\underbrace{\Phi(-0.23506)}_{0.40709} - 120e^{-0.02}\underbrace{\Phi(-0.43506)}_{0.33176} = 5.76$$

Ideally, you should apply some interpolation when using the normal tables to find d_1 and d_2 to ensure that your answers are sufficiently accurate. The method we have used here is to say that $\Phi(0.23506)$ lies between the tabulated values $\Phi(0.23) = 0.59095$ and $\Phi(0.24) = 0.59483$. In fact, because of the third, fourth and fifth digits (506), it lies 50.6% of the way towards the higher value. So a more accurate, interpolated value is:

$$0.506 \times \Phi(0.24) + 0.494 \times \Phi(0.23) = 0.506 \times 0.59483 + 0.494 \times 0.59095 = 0.59291$$

Therefore $\Phi(-0.23506) = 1 - \Phi(0.23506) = 1 - 0.59291 = 0.40709$

This is lower than the stated option price of 10.09, so the implied volatility must be higher than 20%.

This is because the value of a standard European call or put option always increases with volatility, ie vega is positive.

If the volatility σ is instead assumed to be 30%, we have:

$$d_1 = \frac{\ln\frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\frac{110}{120} + (0.02 + \frac{1}{2} \times 0.3^2) \times 1}{0.3\sqrt{1}} = -0.07337$$

and $d_2 = d_1 - \sigma\sqrt{T} = -0.07337 - 0.3\sqrt{1} = -0.37337$

$$\Rightarrow \quad c_0 = 110\underbrace{\Phi(-0.07337)}_{0.47076} - 120e^{-0.02}\underbrace{\Phi(-0.37337)}_{0.35444} = 10.09$$

This is exactly the stated option price, so the implied volatility must be 30% *pa* to the nearest 1%.

If our second try had given us an answer that was too high, we would have used linear interpolation to find a more accurate implied volatility. Or, if our second try had given us an answer that was still too low, we would have tried a third value before being able to use linear interpolation (assuming we weren't running out of time by this stage).

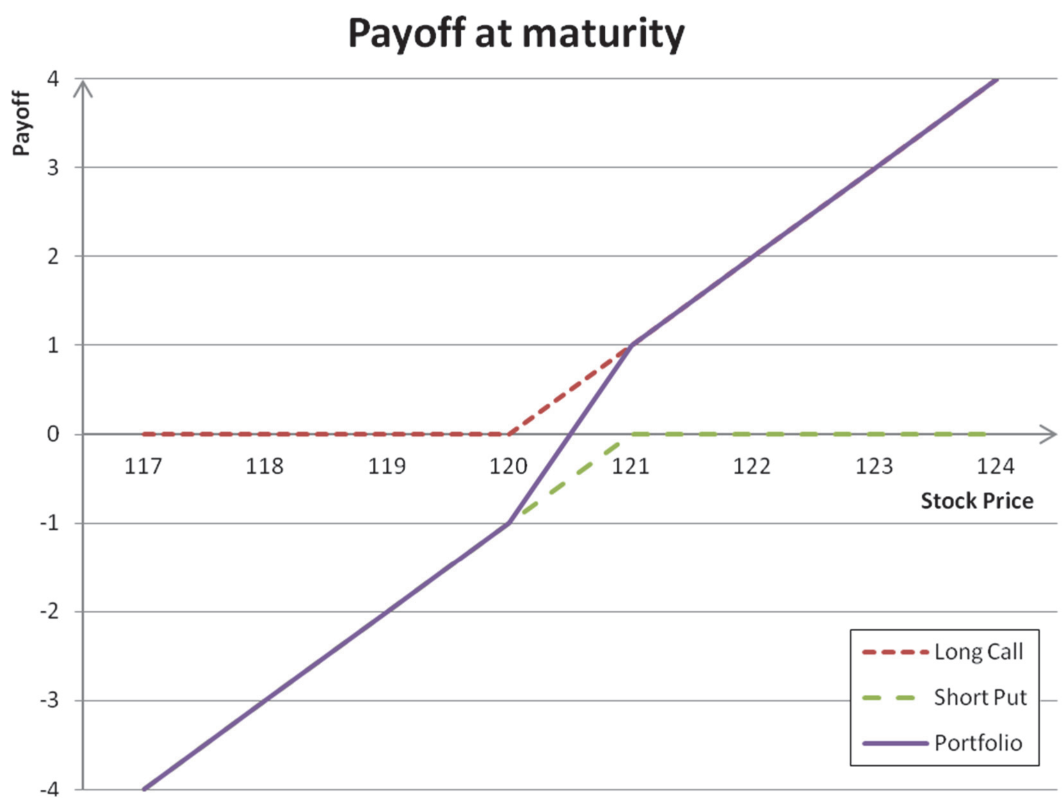
(iii) ***Payoff graph***

We can generate the payoff graph for the portfolio by considering the payoff from each constituent in turn and then adding them together.

The payoff from the long call is $\max\{S_1 - 120, 0\}$.

The payoff from the short put is $-\max\{121 - S_1, 0\}$.

Therefore the portfolio payoff is given by $\max\{S_1 - 120, 0\} - \max\{121 - S_1, 0\}$.



(iv)(a) ***Upper and lower bounds for the value of the portfolio at maturity***

> We can consider the payoff for each straight-line section of the graph separately and see what this produces.
>
> Remember the portfolio payoff is given by $\max\{S_1 - 120, 0\} - \max\{121 - S_1, 0\}$.

The payoff function can be written as:

$$\text{Payoff} = \begin{cases} 0 - (121 - S_1) & = S_1 - 121 & \text{if } S_1 < 120 \\ (S_1 - 120) - (121 - S_1) & = 2S_1 - 241 & \text{if } 120 \le S_1 < 121 \\ (S_1 - 120) - 0 & = S_1 - 120 & \text{if } S_1 \ge 121 \end{cases}$$

Considering these in the context of the graph from part (iii), we can see that the upper and lower bounds are:

Upper bound = $S_1 - 120$

Lower bound = $S_1 - 121$

> *So* $S_1 - 121 \le V_1 (= c_1 - p_1) \le S_1 - 120$ *, using the usual notation.*

(iv)(b) ***Upper and lower bounds for the current put price***

We know from part (iii)(b) that:

$$S_1 - 121 \le c_1 - p_1 \le S_1 - 120$$

Taking the value of each component at time 0 gives:

$$\underbrace{S_0}_{110} - 121e^{-0.02} \le \underbrace{c_0}_{10.09} - p_0 \le \underbrace{S_0}_{110} - 120e^{-0.02}$$

ie: $\underbrace{110 - 118.60}_{-8.60} \le 10.09 - p_0 \le \underbrace{110 - 117.62}_{-7.62}$

ie: $7.62 \le p_0 - 10.09 \le 8.60$

> *Since* c_0 *is given to two decimal places it doesn't make sense to keep more than this for the bounds.*

Rearranging this gives:

$$17.71 \le p_0 \le 18.69$$

(v) ***Fair price of the put***

Since call and put options have different strike prices, we can't use the put-call parity relationship from part (i) to obtain the put price, so we need to use the Black-Scholes option pricing formula for a put option, which can be found on page 47 of the Tables.

The fair price of the put at time 0 is given by:

$$p_0 = Ke^{-rT}\Phi(-d_2) - S_0\Phi(-d_1)$$
$$= 121e^{-0.02}\Phi(-d_2) - 110\Phi(-d_1)$$

So $$d_1 = \frac{\ln\frac{S_0}{K} + (r + \frac{1}{2}\sigma^2)T}{\sigma\sqrt{T}} = \frac{\ln\frac{110}{121} + (0.02 + \frac{1}{2}\times 0.3^2)\times 1}{0.3\sqrt{1}} = -0.10103$$

and $$d_2 = d_1 - \sigma\sqrt{T} = -0.10103 - 0.3\sqrt{1} = -0.40103$$

$$\Rightarrow \quad p_0 = 121e^{-0.02}\times \underbrace{\Phi(0.40103)}_{0.65580} - 110\times\underbrace{\Phi(0.10103)}_{0.54024} = 18.35$$

Solution 9

Overview

The Core Reading on this topic has been reduced significantly since 2014, and so this question would probably now be worth only around 4 marks rather than the original 8.

Empirical tests of the lognormal model (including the normality assumption) are covered in Chapter 11.

Evidence against normality assumptions in models of market returns

There is some empirical research that questions the use of the normality assumptions in market returns.

In particular, *market crashes* appear more often than would be expected from a normal distribution. While the lognormal model produces continuous price paths, jumps or discontinuities seem to be an important feature of real markets.

Furthermore, *days with no change*, or very small change, also happen more often than the normal distribution suggests. This would seem to justify the consideration of Levy processes.

Other distributions can be used, whose parameters have been chosen to give the best fit to the data. However, the improved fit to empirical data comes at the cost of losing the tractability of working with normal (and lognormal) distributions.

Solution 10

The question starts with some straightforward bookwork, and them moves on to become slightly more involved, testing whether you understand the bond pricing formula.

The two-state model for credit ratings is covered in Chapter 19.

(i) ***Formula for general zero-coupon bond price***

It's worth understanding how to build up this formula bit by bit, rather than blindly learning it, as it's easy to get muddled with all the brackets.

The price at time t of a zero-coupon bond that matures at time T with a payoff of 1 is given by:

$$B(t,T)=e^{-r(T-t)}\left[1-(1-\delta)\left\{1-\exp\left(-\int_t^T \lambda(s)ds\right)\right\}\right]$$

where:

- r is the constant continuously-compounded risk-free rate
- δ is the recovery rate
- $\lambda(s)$ the deterministic risk-neutral default intensity.

If we allow $\lambda(s)$ to vary stochastically over time, then we would have:

$$B(t,T)=e^{-r(T-t)}E_Q\left[1-(1-\delta)\left\{1-\exp\left(-\int_t^T \lambda(s)ds\right)\right\}\middle| F_t\right]$$

where Q is the risk-neutral probability measure. This more general case would have been given equal credit in the exam.

(ii) ***Risk-neutral probability of default for Bond A***

This is where it helps to understand where the formula in part (i) comes from. The required probability is represented by the expression in curly brackets. This type of expression might be familiar to you from Subject CT4, which covered survival probabilities.

Let ρ_A be the risk-neutral probability of default for Bond A. Then the current price of Bond A is given by:

$$A(0,1)=e^{-r}\left[1-(1-\delta)\rho_A\right]$$

Rearranging this gives:

$$\rho_A=\frac{1-e^r A(0,1)}{1-\delta}=\frac{1-e^{0.025}\times 0.925}{1-0.5}=0.10317\,,\quad \textit{ie } 10.317\%$$

(iii) ***Formula for the current price of Bond C***

We need to consider the cases where Bond A does or doesn't default separately.

Setting $t=0$, the formula from part (i) can be written in terms of the default probability, ρ, to give:

$$B(0,T)=e^{-rT}\left[1-(1-\delta)\rho\right]$$

Let ρ_C be the risk-neutral probability that Bond C defaults in the second year, conditional on Bond A not defaulting.

Then the current price of Bond C is given by:

$$C(0,2)=e^{-0.025\times 2}\left[1-\underbrace{(1-0)\rho_A}_{(1)}-\underbrace{(1-0.5)(1-\rho_A)\rho_C}_{(2)}\right]$$

(1) represents having a zero recovery rate if Bond A defaults and (2) represents having a 50% recovery rate if Bond A doesn't default, but Bond C does.

This can be simplified to:

$$C(0,2)=e^{-0.05}\left(1-\rho_A\right)\left(1-0.5\rho_C\right)$$

Alternatively, this expression could be written in terms of the default intensities using:

$$\rho_A=\Pr(\textit{Bond A defaults})=1-\exp\left(-\int_0^1\lambda(s)ds\right)$$

$$\rho_C=\Pr(\textit{Bond C defaults}\,|\,\textit{Bond A doesn't default})=1-\exp\left(-\int_1^2\lambda(s)ds\right)$$

ie:
$$C(0,2)=e^{-0.05}e^{-\int_0^1\lambda(s)ds}\left(1-0.5\left(1-e^{-\int_1^2\lambda(s)ds}\right)\right)$$

$$=e^{-0.05}e^{-\int_0^1\lambda(s)ds}\left(0.5+0.5e^{-\int_1^2\lambda(s)ds}\right)$$

$$=0.5e^{-0.05}\left(e^{-\int_0^1\lambda(s)ds}+e^{-\int_0^2\lambda(s)ds}\right)$$

(iv) ***The risk-neutral probability of default for Bond C***

This required probability can be found using the formula from part (iii). Care is needed to get the overall (ie unconditional) probability of default for Bond C rather than the probability conditional on Bond A not defaulting.

Rearranging the final formula from part (iii) gives:

$$(1-0.5\rho_C)=\frac{C(0,2)e^{0.05}}{1-\rho_A}=\frac{0.7472\times e^{0.05}}{1-0.10317}=0.87587$$

$\Rightarrow \quad \rho_C = 2(1-0.87587) = 0.24825$

This represents a conditional probability. To get the required unconditional probability of Bond C defaulting, we need to consider both:

(1) the possibility of Bond A defaulting (in which case Bond C does too), and

(2) the possibility of Bond A not defaulting but Bond C still defaulting.

So the required probability is given by:

$$\begin{aligned}\rho_A + (1-\rho_A)\rho_C &= 0.10317 + (1-0.10317)\times 0.24825 \\ &= 0.32580\end{aligned}$$

ie 32.58

From Subject CT6

Solution 11

This is a very non-standard question on ruin theory ("lowest probability of ... running out of money"). However, since there are so few numbers we're going to have to resort to "common sense" rather than calculation.

It is no surprise that the examiners stated that many candidates did not attempt this question.

The concept is sort of covered by Chapter 20.

With A we start with 10 and each day we subtract the average of 2 and an "error" of ε_i. So the money left each day will be:

Day 1 $10-(2+\varepsilon_1)=8-\varepsilon_1$

Day 2 $(8-\varepsilon_1)-(2+\varepsilon_2)=6-\varepsilon_1-\varepsilon_2$

Day 3 $(6-\varepsilon_1-\varepsilon_2)-(2+\varepsilon_3)=4-\varepsilon_1-\varepsilon_2-\varepsilon_3$

Day 4 $(4-\varepsilon_1-\varepsilon_2-\varepsilon_3)-(2+\varepsilon_4)=2-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4$

Day 5 $(2-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)-(2+\varepsilon_5)=-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5$

Whereas with B we receive 2 each day so we will have:

Day 1 $2-(2+\varepsilon_1)=-\varepsilon_1$

Day 2 $-\varepsilon_1+2-(2+\varepsilon_2)=-\varepsilon_1-\varepsilon_2$

Day 3 $-\varepsilon_1-\varepsilon_2+2-(2+\varepsilon_3)=-\varepsilon_1-\varepsilon_2-\varepsilon_3$

Day 4 $(-\varepsilon_1-\varepsilon_2-\varepsilon_3)+2-(2+\varepsilon_4)=-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4$

Day 5 $(-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)+2-(2+\varepsilon_5)=-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5$

And finally with C we have a discount card and receive 1 each day so will have:

Day 1 $1-(1+\varepsilon_1)=-\varepsilon_1$

Day 2 $-\varepsilon_1+1-(1+\varepsilon_2)=-\varepsilon_1-\varepsilon_2$

Day 3 $-\varepsilon_1-\varepsilon_2+1-(1+\varepsilon_3)=-\varepsilon_1-\varepsilon_2-\varepsilon_3$

Day 4 $(-\varepsilon_1-\varepsilon_2-\varepsilon_3)+1-(1+\varepsilon_4)=-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4$

Day 5 $(-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4)+1-(1+\varepsilon_5)=-\varepsilon_1-\varepsilon_2-\varepsilon_3-\varepsilon_4-\varepsilon_5$

So we can see that B and C are equivalent in terms of the likelihood of ruin, whereas the bigger initial surplus for A means they are more likely to be ruined later.

B and C are equivalent in terms of when ruin is likely to occur. For example, a fare of £2.10 on the first day will cause ruin under both

The larger capital buffer under option A means that they are more likely to last longer before they are ruined, if at all. For example, a fare of £2.10 on the first day will not cause ruin.

So A results in the lowest probability of running out of cash.

Solution 12

Overview

This question involves calculating the probability of ruin using a normal approximation.

Probability of ruin and premium loadings are covered in Chapter 20.

We require the premium loading such that:

$$P[U(1)<0]<0.1$$

Now recall that the surplus/cashflow process is given by:

$$U(t) = u + ct - S(t)$$

where c *, the premium per unit time, is given by* $c = (1+\theta)\lambda E(X)$

u *, is the initial surplus*

$S(t)$ *is the aggregate claim amount up to time t*

Since:

$$E(X) = 0.3 \times 50 + 0.5 \times 100 + 0.2 \times 200 = 105$$

the premium per unit time is given by:

$$c = (1+\theta)\lambda E(X) = (1+\theta) \times 25 \times 105 = 2{,}625(1+\theta)$$

So the surplus process at time 1 will be:

$$U(1) = 240 + 2{,}625(1+\theta) - S(1)$$

Hence, we require:

$$P(240 + 2{,}625(1+\theta) - S(1) < 0) < 0.1$$

$$P(S(1) > 240 + 2{,}625(1+\theta)) < 0.1$$

Now $S(1)$ *has a compound Poisson distribution:*

$$S(1) = X_1 + \cdots + X_{N(1)}$$

where $N(1) \sim Poisson(25)$ *.*

Now $S(1)$ *doesn't have any standard distribution in the Tables. Given that the claim amounts are discrete we could systematically work through each of the possibilities. But the question asks us to calculate* ***approximately*** *the smallest premium. So this means we can make the standard assumption that:*

$$S(1) \sim N(E(S(1)), \text{var}(S(1))) .$$

Using the compound distribution formulae on page 16 of the Tables, we have:

$$E(S(1)) = \lambda E(X)$$

$$\text{var}(S(1)) = \lambda E(X^2)$$

For our claims distribution we have:

$$E(X^2) = 0.3 \times 50^2 + 0.5 \times 100^2 + 0.2 \times 200^2 = 13{,}750$$

So, we have:

$$E[S(1)] = \lambda E(X) = 25 \times 105 = 2,625$$

$$\text{var}[S(1)] = \lambda E(X^2) = 25 \times 13,750 = 343,750$$

Alternatively, we could have calculated $\text{var}(X) = 2,725$ *and then used the standard compound distribution formulae:*

$$E(S) = E(N)E(X)$$

$$\text{var}(S) = E(N)\text{var}(X) + \text{var}(N)[E(X)]^2$$

Using a normal approximation:

$$S(1) \sim N(2625, 343750)$$

Hence:

$$P(S(1) > 240 + 2,625(1+\theta)) < 0.1$$

$$P\left(Z > \frac{240 + 2,625(1+\theta) - 2,625}{\sqrt{343,750}}\right) < 0.1$$

$$P\left(Z > \frac{240 + 2,625\theta}{\sqrt{343,750}}\right) < 0.1$$

Technically, since the claims are discrete we should have used a continuity correction. The examiners gave students marks regardless of whether one was used or not.

From page 162 of the *Tables*, we see that $P(Z > 1.2816) = 0.1$. Hence, we require that:

$$\frac{240 + 2,625\theta}{\sqrt{343,750}} > 1.2816$$

$$\theta > \frac{1.2816\sqrt{343,750} - 240}{2,625} = 0.1948$$

So we require the premium loading factor to be greater than 19.48%.

This premium loading factor gives the probability of ruin in the first year as 10%. Larger premium factors would reduce the probability of ruin to less than 10%. Hence, we have calculated the smallest premium factor as required.

Solution 13

Overview

This question tests the inflation-adjusted chain ladder method is described in Chapter 21. However, it gives us an inflation index rather than the annual rates of inflation.

We start by adjusting the data for past inflation. Inflation adjustment must be applied to incremental (rather than cumulative) data, but note that we are given the incremental data here.

The values on the main diagonal are already at 2013 prices, so they do not need to be adjusted.

The values of 1,718 and 535 on the diagonal above that are at 2012 prices. So they need to be multiplied by $\frac{110}{107}$ *to adjust them to 2013 prices.*

Finally, the value of 1,403 is at 2011 prices, so it needs to be multiplied by $\frac{110}{100}$ *to adjust it to 2013 prices.*

Adjusting for past inflation gives us the following incremental claims at 2013 prices:

		Development year		
		0	1	2
Accident year	2011	$1,430\times\frac{110}{100}=$ **1,543.3**	$535\times\frac{110}{107}=$ **550**	142
	2012	$1,718\times\frac{110}{107}=$ **1,766.17**	811	
	2013	1,912		

The cumulative figures at 2013 prices are:

		Development year		
		0	1	2
Accident year	2011	1,543.3	2,093.3	2,235.3
	2012	1,766.17	2,577.17	
	2013	1,912		

We now calculate the development factors. They are:

Year 0 to Year 1: $\frac{2,093.3+2,577.17}{1,543.3+1,766.17}=1.411244$

Year 1 to Year 2: $\frac{2,235.3}{2,093.3}=1.067835$

The projected cumulative figures at 2013 prices are:

		Development year		
		0	1	2
Accident year	2011			
	2012			2,751.99
	2013		2,698.30	2,881.34

The projected incremental figures at 2013 prices are:

		Development year		
		0	1	2
Accident year	2011			
	2012			174.82
	2013		786.30	183.04

The figures of 786.30 and 174.82 need to get multiplied by $\frac{113}{110}$ *to bring them up to 2014 prices, and the figure of 183.04 needs to get multiplied by* $\frac{117}{110}$ *to bring it up to 2015 prices.*

The projected incremental figures adjusted for future inflation are:

		Development year		
		0	1	2
Accident year	2011			
	2012			$174.82\times\frac{113}{110}=$ **179.59**
	2013		$786.30\times\frac{113}{110}=$ **807.74**	$183.04\times\frac{117}{110}=$ **194.69**

The estimated reserve is the sum of these projected figures:

$$179.59+807.74+194.69=1{,}182.02$$

Note the question doesn't state any units so you shouldn't be writing £1,182.02.